Unambiguity in Transducer Theory

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Dagstuhl Seminar on Unambiguity

WHAT IS A (ONE-WAY) TRANSDUCER?

TRANSDUCERS are automata with output.

$\mathcal{A} = (\Sigma, Q, q_0, F, \Delta) \leftarrow$ NFA

Monoid

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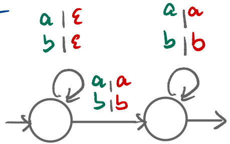
NFT



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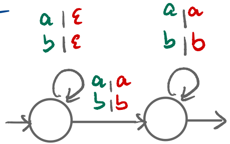

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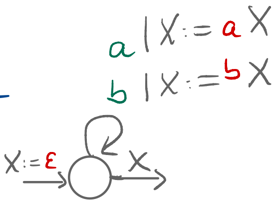

STREAMING STRING TRANSDUCER

Alur, Černý '10

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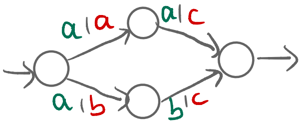
EXAMPLES

- unambiguous



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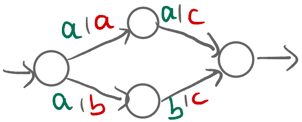

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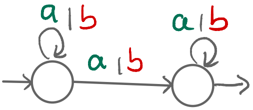

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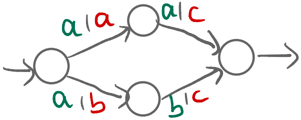

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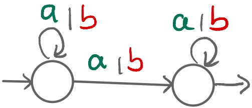
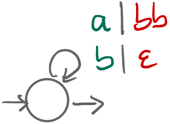


- sequential (= input-deterministic)



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Thm Every function recognized by a transducer $T = (A, \sigma : \Delta \to M)$ can be recognized by an unambiguous one.

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- functional SUM-automata ≡ unambiguous SUM-automata

  $M = (\mathbb{Z}, +)$

MORE GENERAL RESULT (UNIFORMISATION PROPERTY)

Thm Every relation recognized by a transducer contains a function with equal domain that can be recognized by an unambiguous transducer.

$\forall R\ \exists f:\ f \subseteq R\ \wedge\ dom(f) = dom(R)$

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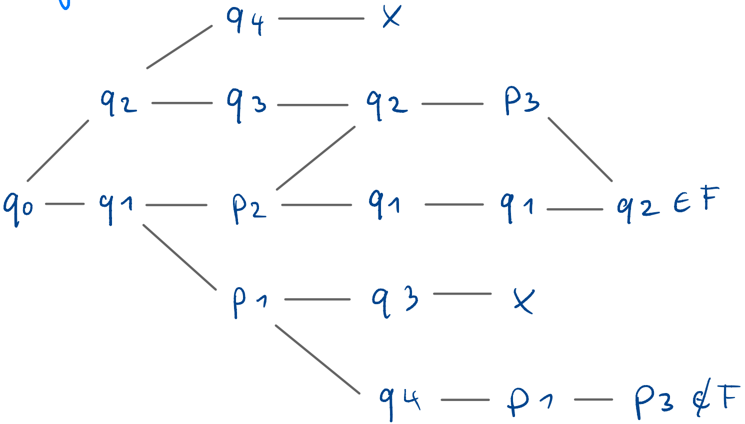
One of many proofs.
(First one: Eilenberg '74)

PROOF

Selecting the smallest run via any fixed ordering on the states.



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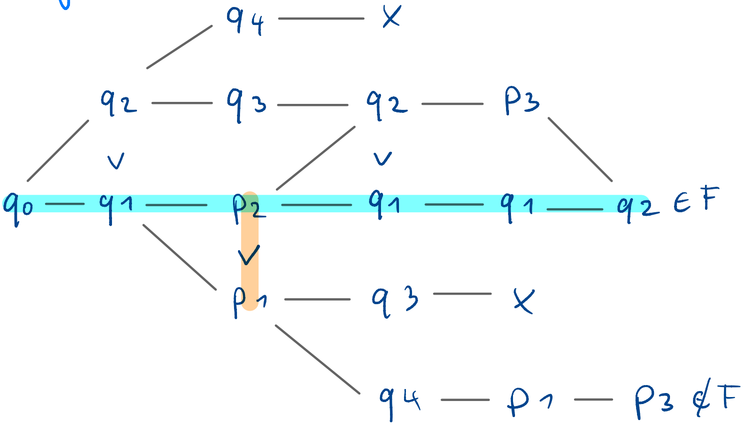


q_0	q_1	p_2	q_1	q_1	$q_2 \in F$
$\emptyset$	$\emptyset$	$\{p_1\}$	$\{q_3, q_4\}$	$\{p_1\}$	$\{p_3\} \cap F = \emptyset$

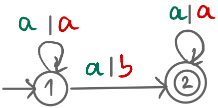
TRANSDUCERS ON INFINITE WORDS

$T = (A, \sigma)$, A with Büchi acceptance.

Thm Every relation recognized by a Büchi-transducer contains a function with equal domain that can be recognized by an unambiguous Büchi-transducer.

Proof Order the runs. Select the smallest accepting run.

⚠ Not every state ordering works ⚠

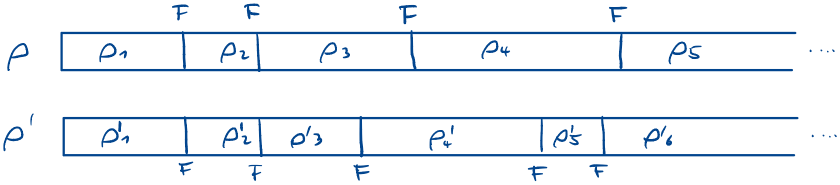


$2 < 1$: For every accepting run there exists a smaller accepting run.

PROOF IDEA: HOW TO ORDER RUNS

Choffrut & Grigorieff '99

Cut runs into segments at final states



- Segment $\rho_i < \rho'_i$ if $\rho_i <_{llex} \rho'_i$ for some fixed ordering on Q

  length-lexicographic

- Run $\rho < \rho'$ if there is some i s.t. $\rho_i < \rho'_i$ and for all $j < i$: $\rho_j = \rho'_j$.

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$\Rightarrow$ fNFT $<_{exp}$ UFT, but

OPEN

$f\text{NFT}_{\substack{\text{total}\\\text{domain}}} <_{exp} \text{UFT}_{\substack{\text{total}\\\text{domain}}}$?

IS UNAMBIGUITY NECESSARY?

EXAMPLES (from $M = \Sigma^*$)

$$f_1 : \underset{n \geq 1}{a^n} \mapsto \begin{cases} a^n & n \text{ odd} \\ b^n & n \text{ even} \end{cases}$$

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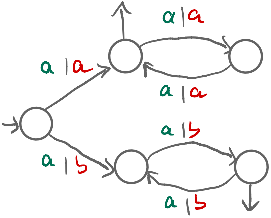

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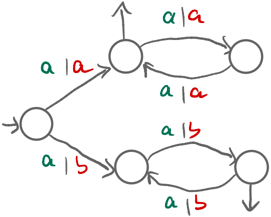

$$f_2 : uba^n \mapsto ub$$

$u \in \{a,b\}^*$

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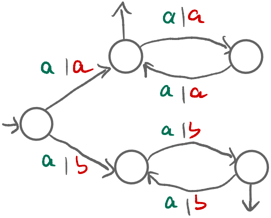

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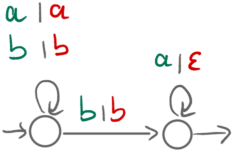

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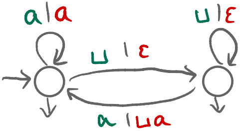

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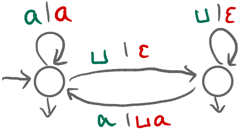

Thm Given a fNFT $\mathcal{T}$, it is decidable in PTIME whether $\mathcal{T}$ is equivalent to some sequential (= input-deterministic) transducer.

Béal, Carton, Prieur, Sakarovitch '03

REMOVING UNAMBIGUITY WITH TWO-WAYNESS

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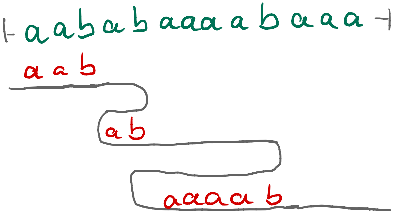

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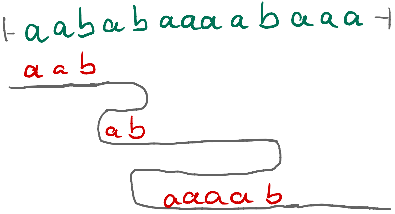

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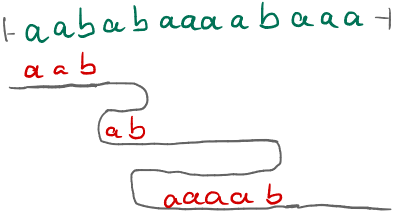

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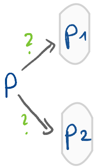

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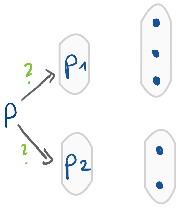

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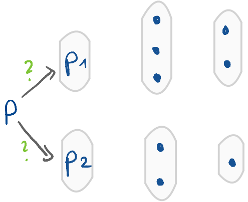

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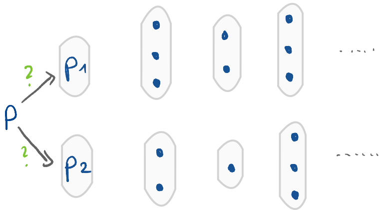

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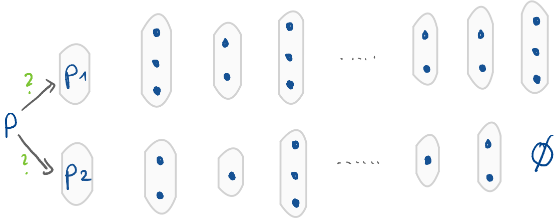
①



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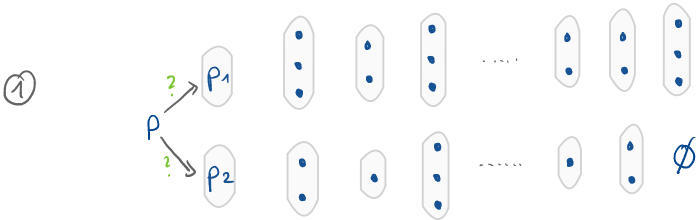

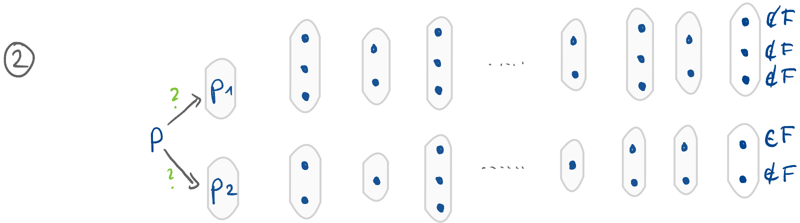

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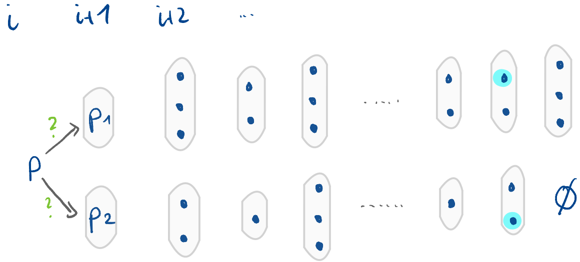

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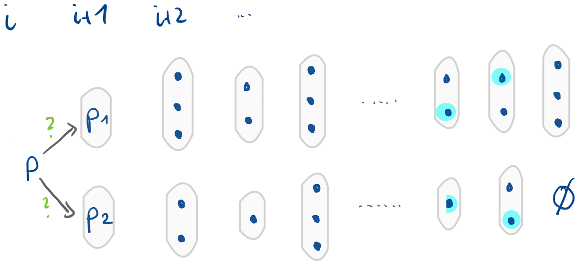

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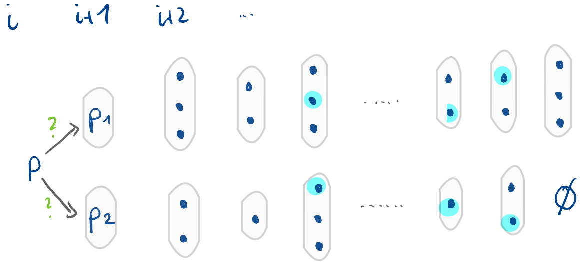

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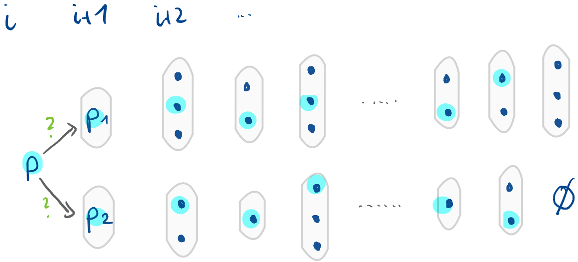

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- $$f = \overset{\text{DET}}{f_2} \circ \overset{\text{coDET}}{f_1} = \overset{\text{DET}}{f_2} \circ \overset{\text{2-DET}}{f_1}$$

- $$\underset{\text{Elgot Mezei}}{\text{RAT} \equiv} \text{DET} \circ \text{coDET} \underset{\text{Hopcroft Ullman}}{\equiv} \text{DET} \circ \text{2DET} \underset{\wedge}{\equiv} \text{2DET}$$

2WAY $q \xrightarrow{\sigma \mid u, d} q'$

DET $p \xrightarrow{u \mid v} p'$

COMP $(q,p) \xrightarrow{\sigma \mid v, d} (q', p')$

SUMMARY

Thm $f\text{NFT} \equiv \text{UFT} \supsetneq$ sequential FTs

$\cap\!\mid$

2DFT

SUMMARY

<u>Thm</u> $f\,\mathrm{NFT} \equiv \mathrm{UFT} \supsetneq$ sequential FTs

$$\begin{array}{c} f\,\mathrm{NFT} \\ \cap\!\mathrm{I} \\ \mathrm{2DFT} \end{array}$$

<u>Thm</u> $f\,\mathrm{SST} \equiv \mathrm{USST}$

SUMMARY & MORE

Thm

$$\begin{array}{l} f\text{NFT} \equiv \text{UFT} \supsetneq \text{sequential FTs} \\ \quad \cap\!\!| \\ \text{2DFT} \end{array}$$

Thm

$$f\,\underbrace{\text{SST} \equiv \text{USST} \equiv \text{DSST} \equiv \text{2DFT}}_{\text{Alur, Černý '10}}$$

SUMMARY & MORE

Thm $\quad f\text{NFT} \equiv \text{UFT} \supsetneq \text{sequential FTs}$

$$\begin{array}{c} f\text{NFT} \\ \cap\!\!\shortmid \\ \text{2DFT} \end{array}$$

Dartois, Fournier, Jecker, Lhote '17

Thm $\quad \underbrace{f\text{SST} \equiv \text{USST} \equiv \text{DSST} \equiv \text{2DFT}}_{\text{Alur, Černý '10}} \equiv \text{2RFT}$

$\boxed{\text{REVERSIBLE} = \text{det. \& co-det.}}$

SUMMARY & MORE

Thm fNFT $\equiv$ UFT $\supsetneq$ sequential FTs
$\cap$
2DFT

Dartois, Fournier, Jecker, Lhote '17

Thm $\underbrace{f\text{SST} \equiv \text{USST} \equiv \text{DSST} \equiv \text{2DFT}}_{\text{Alur, Černý '10}} \equiv$ 2RFT

REVERSIBLE
= det. & co-det.

Thm Every relation recognizable by a 2NFT contains a function with equal domain that can be recognized by a
| 2DFT De Souza '13
| 2RFT Dartois, Fournier, Jecker, Lhote '17

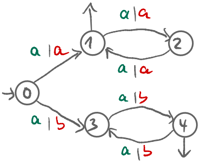
FUNCTION CONCATENATION

Thm Every rational function f can be expressed as $f = f_2 \circ f_1$, f_2 is left-sequential, f_1 is right-sequential. Elgot, Mezei '65

right-seq. (co-det. 1FT): on input $u = u_1 u_2 \dots u_n$ the ith output letter $= \{ (p,q) \mid A : p \xrightarrow{u_i \dots u_n} q \in F \}$

a	a	a	a	a	a	
(1,1)	(0,1)	(1,1)	(0,1)	(1,1)	(0,1)	(1,1)
(0,4)	(2,1)	(0,4)	(2,1)	(0,4)	(2,1)	(4,4)
(4,4)	(3,4)	(4,4)	(3,4)	(0,4)	(3,4)	

$\mathcal{T} = (A, o)$ unambiguous



left-sequential

(1,1)	(0,1)	(1,1)	(0,1)	(1,1)	(0,1)	(1,1)
(0,4)	(2,1)	(0,4)	(2,1)	(0,4)	(2,1)	(4,4)
(4,4)	(3,4)	(4,4)	(3,4)	(0,4)	(3,4)	
ε	b	b	b	b	b	b