Synthesis of Computable Functions

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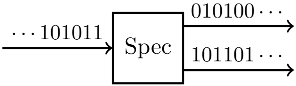
Reactive synthesis of non-terminating systems

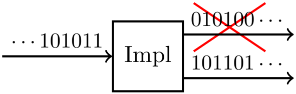
synthesize

Specification - - - - - - - - - - → **Implementation**

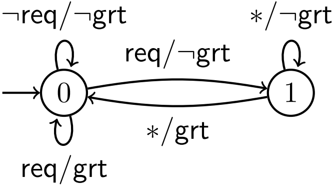
one input is in relation with several outputs

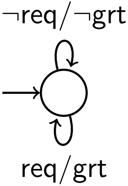
unique output for each input





Church synthesis





Synchronous specifications
(**automatic relations**)
e.g, given by
synchronous transducers with
parity acceptance

Synchronous implementations
given by
Mealy machines

Theorem (Büchi/Landweber'69).
The Church synthesis problem is decidable.

General problem

Goal Decide whether a specification is implementable.

Example.

- Specification: contains pairs of the form

$$(a_1 a_2 a_3 \cdots, a_3 \cdots) \in \{a, b\}^\omega \times \{a, b\}^\omega$$

- no implementation by a Mealy machine exists, but can be implemented

General problem

Goal Decide whether a specification is implementable.

Example.

- Specification: contains pairs of the form

$$(uA\alpha, Au\beta) \quad (uB\alpha, Bu\beta),$$

  where $u\alpha, \beta \in \{a, b\}^\omega$, and A, B are special letters
- can be implemented, e.g., by a deterministic machine that computes the function

$$uA\alpha \mapsto Au\alpha \quad uB\alpha \mapsto Bu\alpha$$

Computability

What does it mean to be **implementable** for a relation?

- There is a computable function f with the same domain as the relation R such that $(\alpha, f(\alpha)) \in R$ for all $\alpha \in \mathrm{dom}(R)$.

A function $f\colon \Sigma^\omega \rightharpoonup \Gamma^\omega$ is **computable** if there exists a deterministic Turing machine M with three tapes,

- a read-only one-way input tape,
- a two-way working tape, and
- a write-only one-way output tape

that works as follows: if the input tape holds an input sequence $\alpha \in \mathrm{dom}(f)$, then

- M outputs longer and longer prefixes of $f(\alpha)$
- when reading longer and longer prefixes of α.

Examples

- $f_1 \colon uA\alpha \mapsto Au\alpha \quad uB\alpha \mapsto Bu\alpha$,
  for all $u\alpha \in \{a,b\}^\omega$, and A, B are special letters

  is computable
- $$f_2 \colon \alpha \mapsto \begin{cases} a^\omega & \text{if } \alpha \text{ contains } \infty \text{ many } a \\ b^\omega & \text{otherwise} \end{cases}$$
  for all $\alpha \in \{a,b\}^\omega$

  is not computable

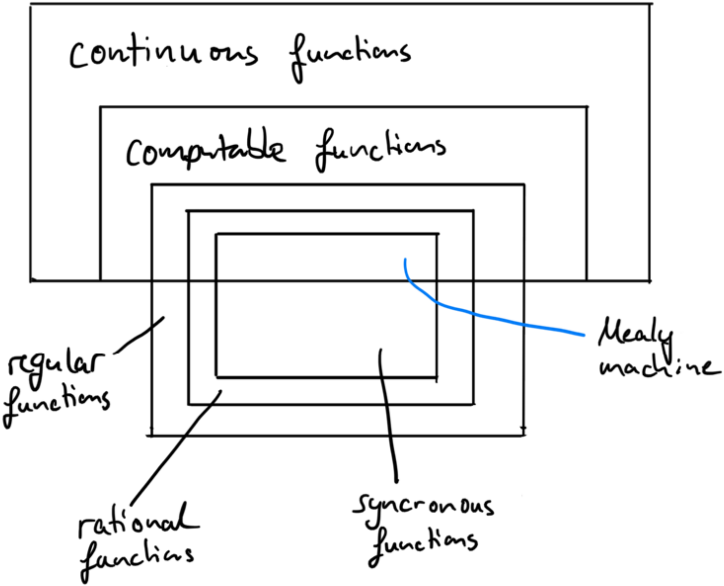
Continuity

Metric (Cantor distance)

Given $\alpha, \beta \in \Sigma^\omega$, distance $d(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha = \beta \\ 2^{-|\alpha \wedge \beta|} & \text{otherwise} \end{cases}$

A **continuous** function $f\colon \Sigma^\omega \rightharpoonup \Gamma^\omega$ ensures that any finite output prefix only depends on a finite input prefix.

Computability and Continuity



Rational relations

Theorem (Filiot/W.'21). Let $R \in \mathsf{RAT}$, it is undecidable whether R is realizable by a continuous or a computable function.

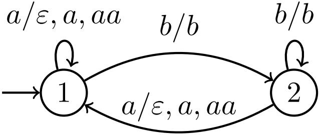
Finite word setting Undecidable whether a rational relation is realizable by a sequential function. (Carayol/Löding'14)

Weakly deterministic rational relations

New Weakly deterministic transducer, defines the class of **weakly deterministic rational relations**.

Theorem (Filiot/W.'21). AUT $\subsetneq$ DRAT $\subsetneq$ wDRAT $\subsetneq$ RAT

Example.



Result

Theorem (Filiot/W.'21). Let $R \in \mathsf{wDRAT}$, it is ExpTime-complete to decide whether R is realizable by a continuous function.

Theorem (Holtmann/Kaiser/Thomas'10 Klein/Zimmermann'14). Let $R \in \mathsf{AUT}$ with total domain, it is ExpTime-complete to decide whether R is realizable by a continuous function.

Equivalence results

Theorem (Filiot/W.'21). Let $R \in \mathsf{wDRAT}$, it is ExpTime-complete to decide whether R is realizable by a continuous function.

Theorem (Filiot/W.'21). Let $R \in \mathsf{wDRAT}$. The following are equivalent:

1. R is realizable by a continuous function
2. R is realizable by a computable function
3. R is realizable by a function computable by a deterministic two-way transducer
4. R is realizable by a continuous rational function
5. R is realizable by a computable rational function

Equivalence results

Theorem (Filiot/W.'21). Let $R \in \mathsf{wDRAT}$, it is ExpTime-complete to decide whether R is realizable by a continuous function.

Theorem (Filiot/W.'21). Let $R \in \mathsf{wDRAT}$ with total domain. The following are equivalent:

1. R is realizable by a continuous function
2. R is realizable by a function computable by a sequential transducer